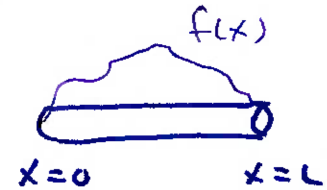
Heat Eq. (2)

last time: $u_t = k u_{xx}$ $\quad 0 < x < L \quad t > 0$



$u(0,t) = 0$ left temp = 0

$u(L,t) = 0$ right temp = 0

$u(x,0) = f(x)$ initial temp. profile

$$u(x,t) = X(x)\,T(t)$$

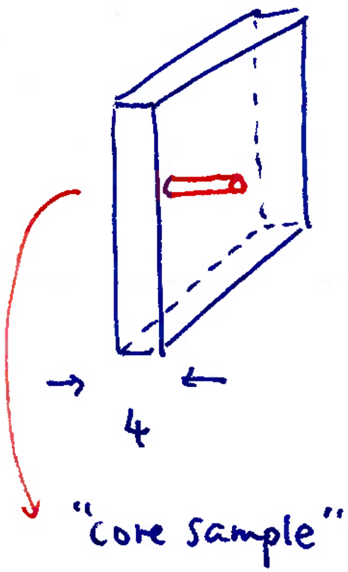
$$X(x) = \sin\left(\frac{n\pi x}{L}\right) \quad \text{spatial solution}$$

$$T(t) = e^{-kn^2\pi^2 t/L^2} \quad \text{temporal solution}$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

$$C_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

<u>example</u> Copper plate/slab thickness 4 cm $\quad K = 1.15\ cm^2/s$



heat the entire interior uniformly to 100 °C at $t=0$

keep right and left faces at 0 °C for all t

entire plate is uniformly heated and made of same material (same K)

→ no lateral heat flow

→ only toward the faces (ends)

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-1.15 n^2 \pi^2 t/16} \sin\left(\frac{n\pi x}{4}\right)$$

initial : $f(x) = 100$

$\vdots$

$$C_n = \frac{2}{4}\int_0^4 100 \sin\left(\frac{n\pi x}{4}\right) dx = \frac{200\left[1-(-1)^n\right]}{n\pi}$$

as $t \to \infty$, the time solution $\to 0$ → heat flows out of rod through ends

space solution → periodic → same temps on ends

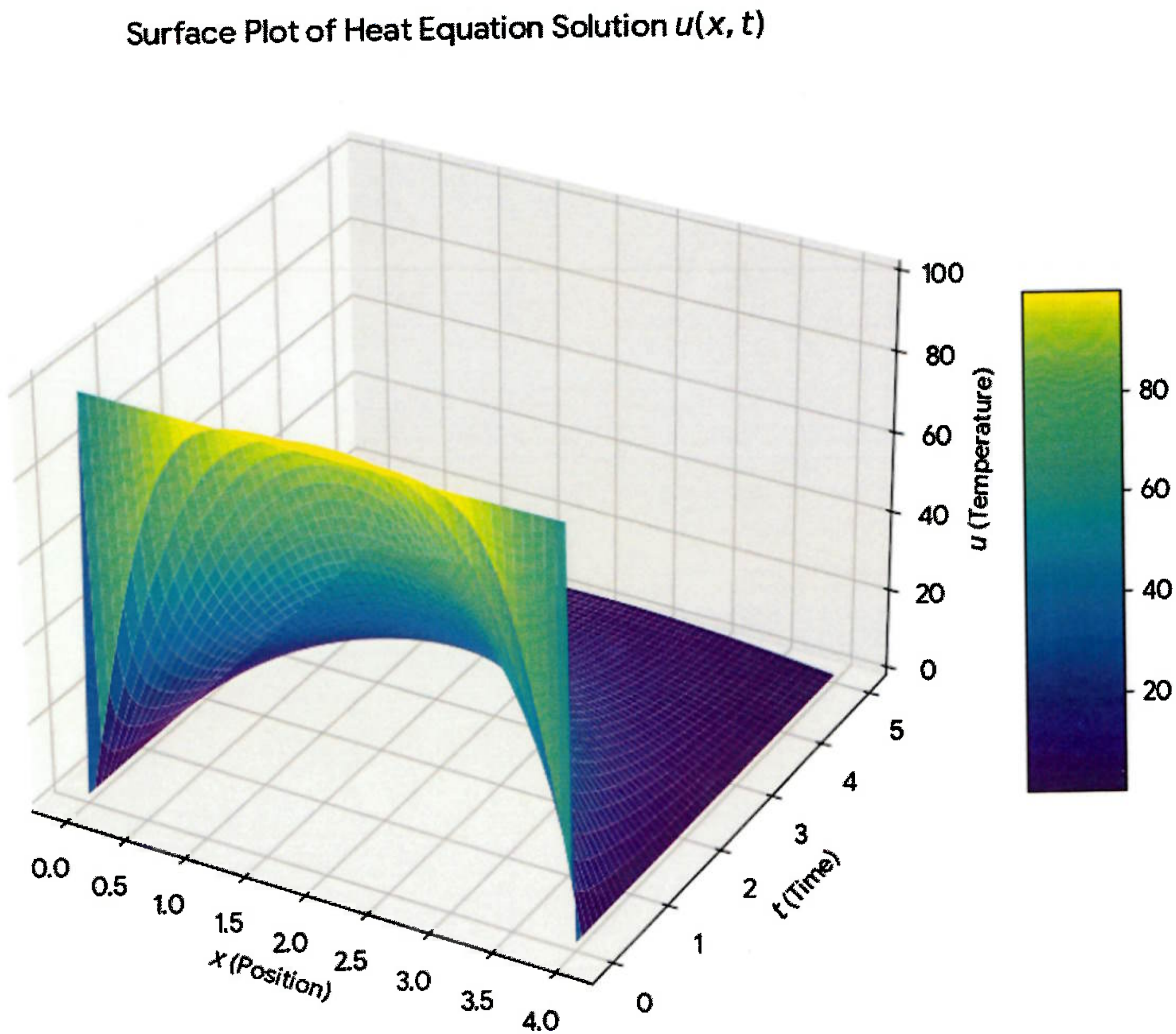
what is the temp at the mid point $(x=2)$ 3 seconds later $(t=3)$

~~$u(x,t) = \sum$~~

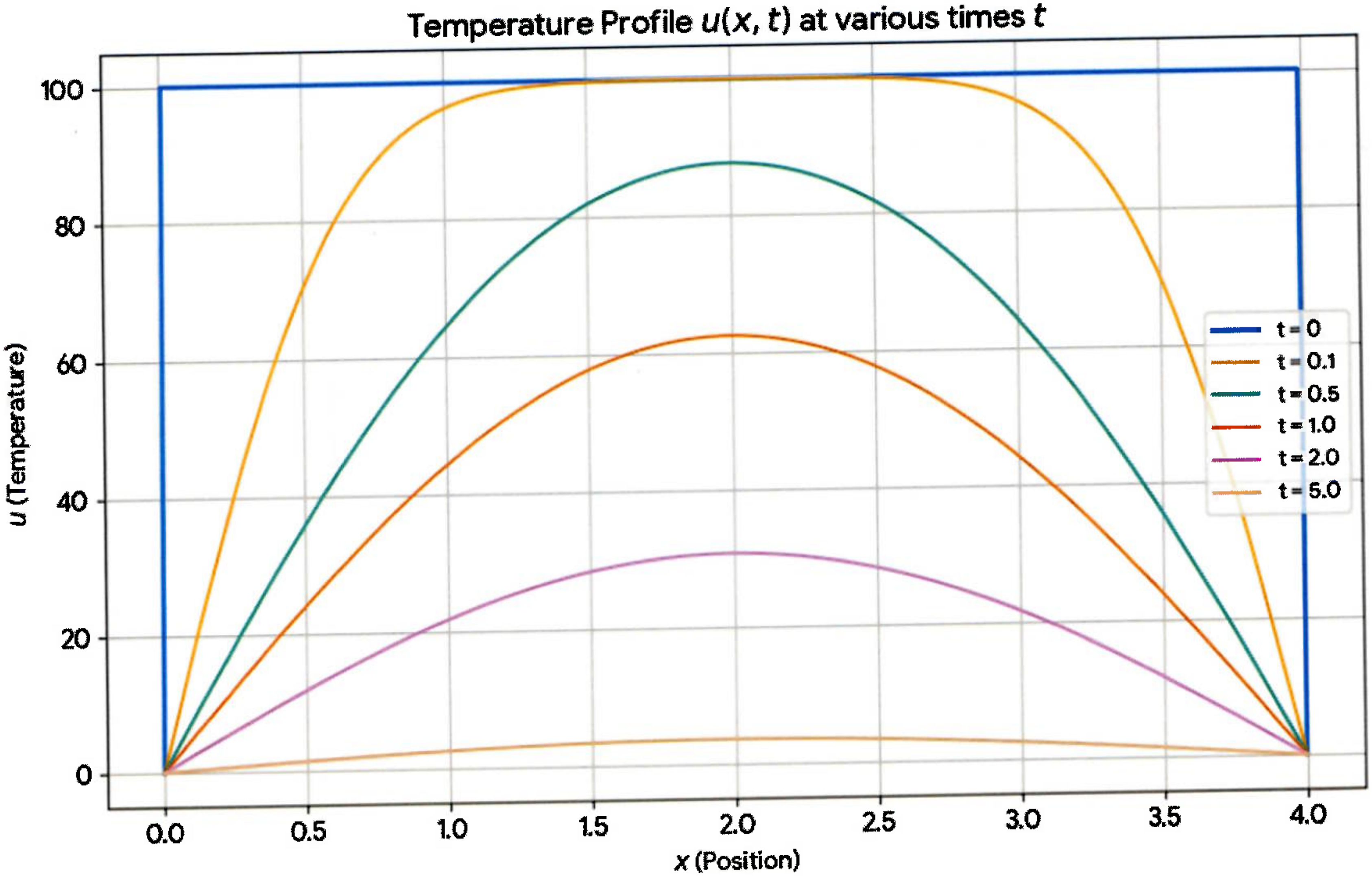
$$u(2,3) = \sum_{n=1}^{\infty} \frac{200[1-(-1)^n]}{n\pi} e^{-1.15 n^2 \pi^2 \cdot 3/16} \sin\left(\frac{n\pi \cdot 2}{4}\right)$$ infinite sum

in practice, the negative exponential makes this a fast converging series → only few terms needed to get "good" approx.

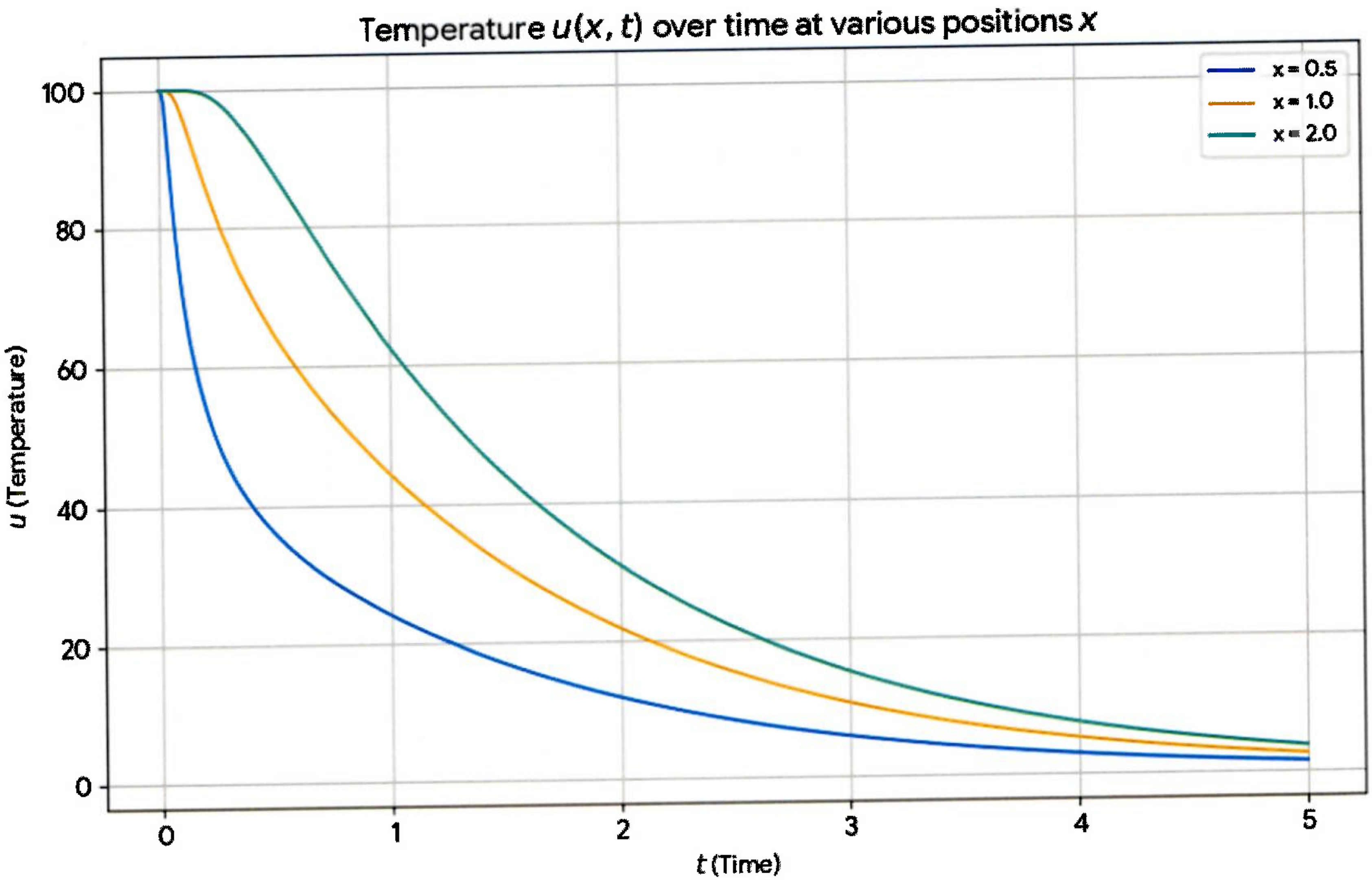
1-term approx ($n=1$ only) → 15.16 °C



Surface Plot of Heat Equation Solution $u(x, t)$
100
80
60
40
20
0
u (Temperature)
80
60
40
20
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
x (Position)
0
1
2
3
4
5
t (Time)



Temperature Profile $u(x, t)$ at various times t
100
80
60
40
20
0
u (Temperature)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
x (Position)
t = 0
t = 0.1
t = 0.5
t = 1.0
t = 2.0
t = 5.0



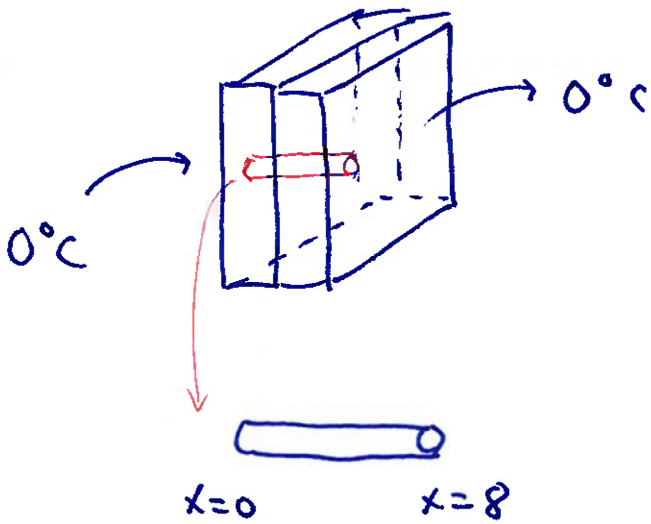
Temperature $u(x, t)$ over time at various positions x
x = 0.5
x = 1.0
x = 2.0
100
80
60
40
20
0
u (Temperature)
0
1
2
3
4
5
t (Time)

two such slabs put together (4 cm thick each)

two outer faces at 0°C

left slab heated to 50°C at t = 0

right slab heated to 100°C at t = 0

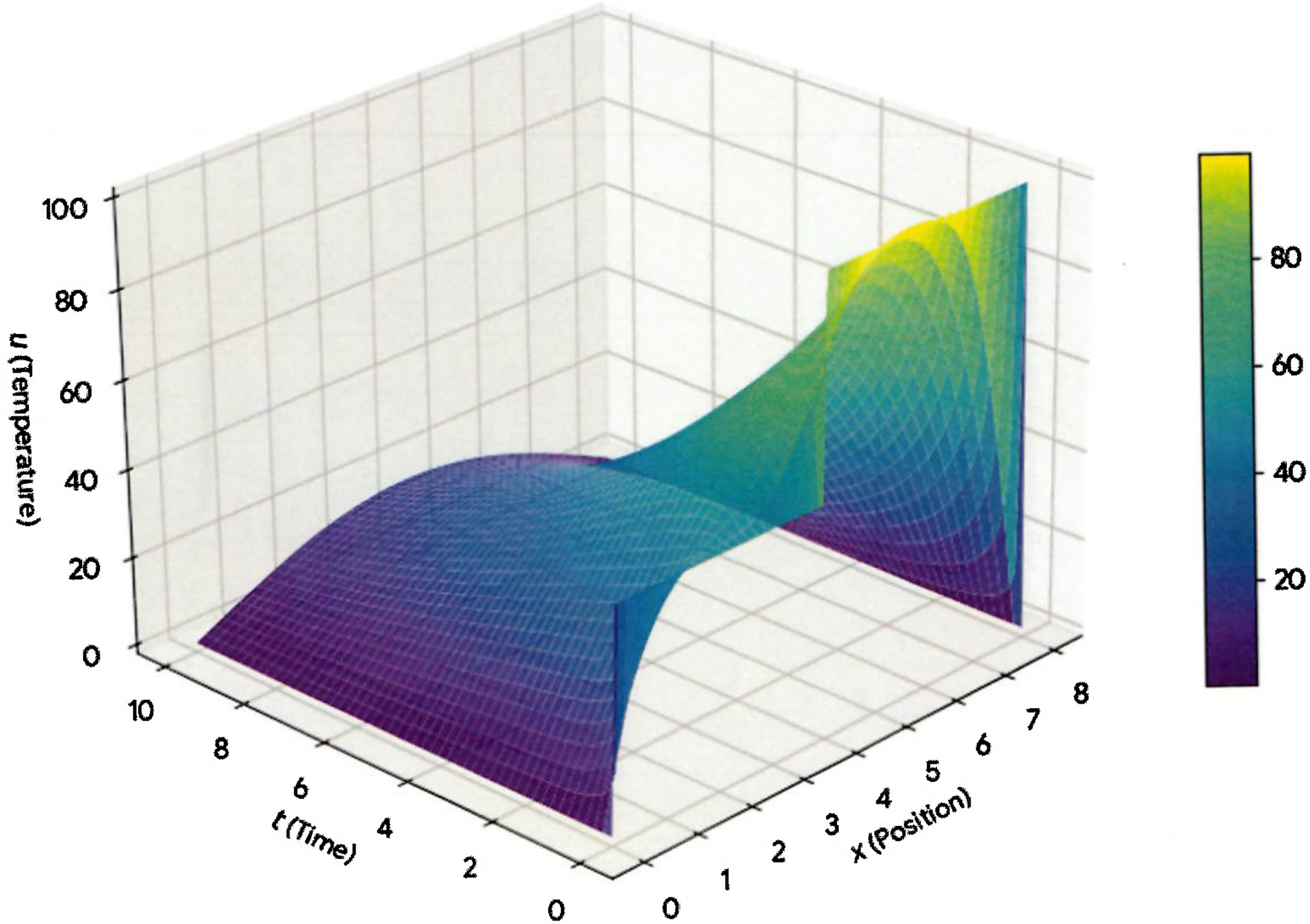


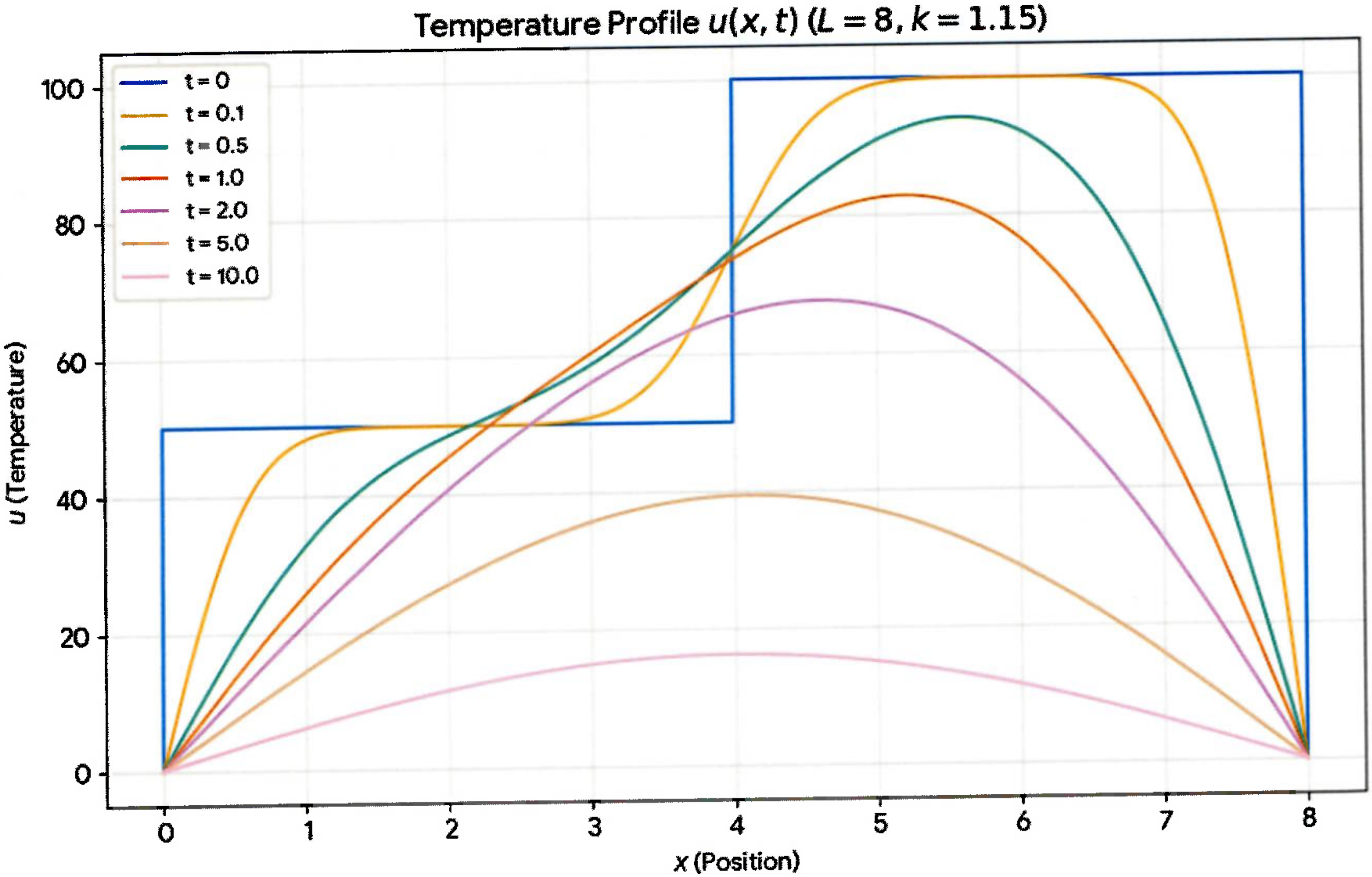
Initial temp: $f(x) = \begin{cases} 50 & 0<x<4 \\ 100 & 4<x<8 \end{cases}$

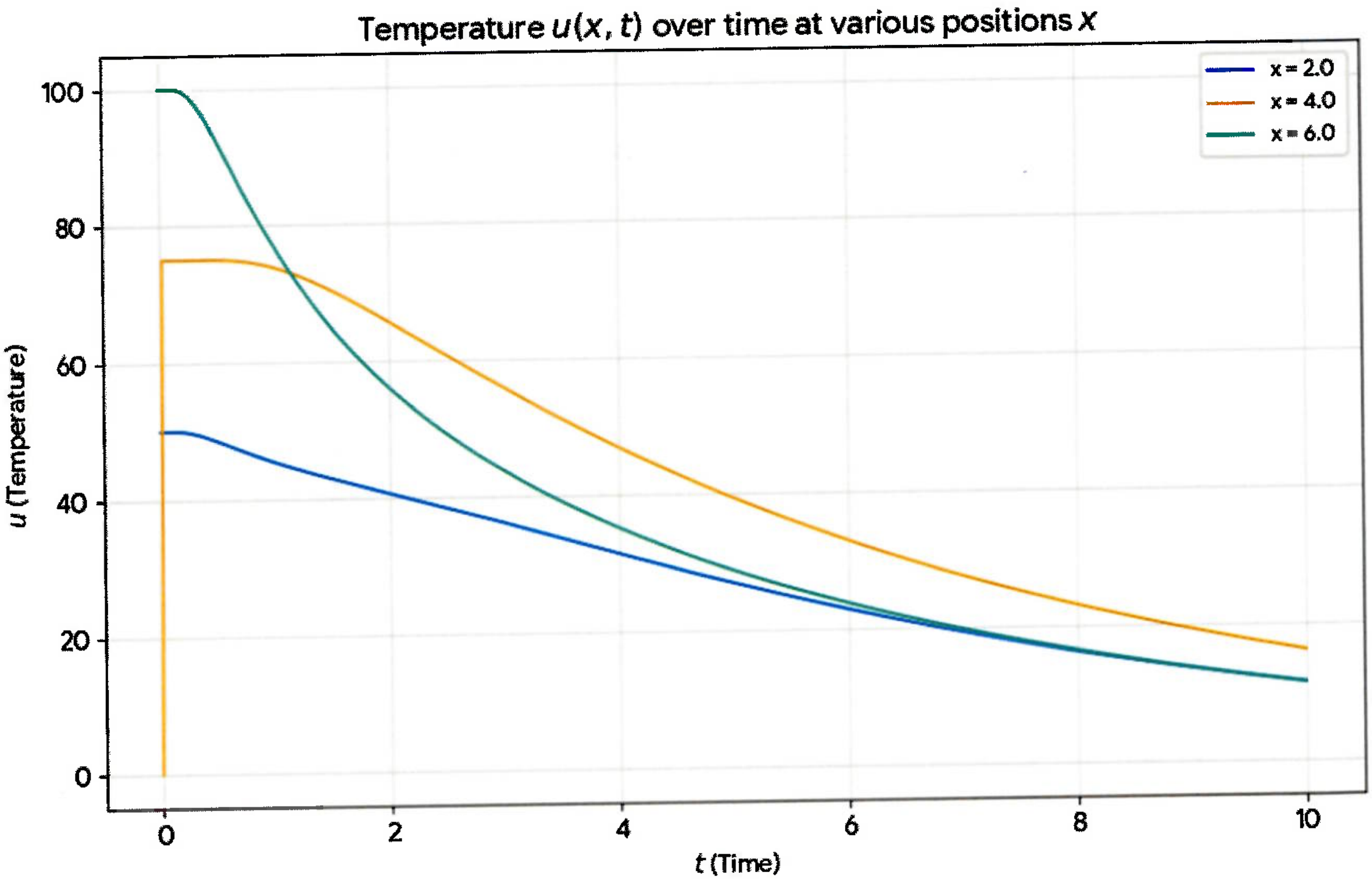
⋮

$$u(x,t) = \sum_{n=1}^{\infty} \frac{100}{n\pi}\left[1+\cos\left(\frac{n\pi}{2}\right)-2(-1)^n\right] e^{-1.15n^2\pi^2 t/64} \sin\left(\frac{n\pi x}{8}\right)$$

Surface Plot $u(x, t)$ ($L = 8$, $k = 1.15$, non-uniform initial)




Temperature Profile $u(x, t)$ ($L = 8$, $k = 1.15$)
t = 0
t = 0.1
t = 0.5
t = 1.0
t = 2.0
t = 5.0
t = 10.0
u (Temperature)
x (Position)
0
20
40
60
80
100
0
1
2
3
4
5
6
7
8



Temperature $u(x, t)$ over time at various positions x
x = 2.0
x = 4.0
x = 6.0
100
80
60
40
20
0
u (Temperature)
0
2
4
6
8
10
t (Time)

now let's relax the ends at 0°C constraint

$u_t = k u_{xx} \quad 0<x<L \quad t>0$

$u(0,t) = T_1$

$u(L,t) = T_2$

if T_1, T_2 are not zero, the BC's are <u>nonhomogeneous</u>

steady-state temp solution

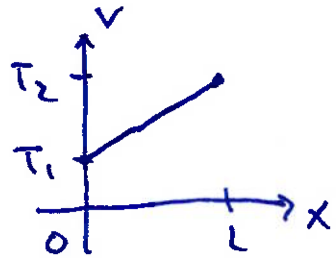
$\longrightarrow$ $t \to \infty$ or $u_t = 0$ (time is not a factor any more)

in $u_t = k u_{xx}$ if $u_t = 0$

$u_{xx} = 0 \rightarrow u = c_1 + c_2 x$ using $u(0) = T_1$, $u(L) = T_2$

we get

$$u = \frac{T_2 - T_1}{L} x + T_1 = V(x) \quad \text{steady-state temp.}$$



but what about the transient solution
(when time still matters)

$u_t = k u_{xx}$ $\qquad$ $v(x) = \frac{T_2 - T_1}{L} x + T_1$

$u(0, t) = T_1$

$u(L, t) = T_2$

$u(x, 0) = f(x)$

define $w(x,t) = u(x,t) - v(x)$

$w_t = u_t \qquad w_{xx} = u_{xx} \rightarrow \quad w_t = k w_{xx}$ same heat eq.

$w(0, t) = 0$

$w(L, t) = 0$

} BC's are homogeneous in w

solution is <u>known</u>

$$w(x,t) = \sum_{n=1}^{\infty} B_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

$$u(x,t) = V(x) + \sum_{n=1}^{\infty} B_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

initial condition : $u(x,0) = f(x)$

$$f(x) = V(x) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) - V(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$ Sine series

$$B_n = \frac{2}{L}\int_0^L \left[f(x) - V(x)\right] \sin\left(\frac{n\pi x}{L}\right) dx$$